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Spherical gravitational collapse with escaping neutrinos

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Abstract. The general relativity equations for the dynamics of spherically symmetric perfect fluid distributions with an outward neutrino flux are studied. The geometrical part of the field equations is simplified by introducing the eigenvalue of the Weyl tensor in the 2–2 component of the Einstein tensor. This eigenvalue is interpreted as the energy density of the *free gravitational field* which contributes to the mass $m(r, t)$ contained within a coordinate radius r of the fluid sphere. It is pointed out that the presence of a free gravitational field produces shearing forces inside the material sphere. An energy equation which shows clearly how the total (fluid and radiation) pressure does work in a material sphere across its moving boundary is obtained.

When the energy density of the free gravitational field vanishes ($\epsilon = 0$), it is shown that the space–time is conformally flat. Taking the interior geometry of a star to be conformally flat and assuming that it is filled with a perfect fluid, the following results are obtained: (i) the streamlines of the fluid are orthogonal to the hypersurfaces $\rho(R, T) =$ constant, (ii) the conservation laws are identities, (iii) the rate of contraction of a fluid sphere $\mathring{U} = 0 \Leftrightarrow \rho =$ constant. It is further shown that a zero rest mass field representing an unpolarized outward neutrino flux can be introduced into the system without disturbing the conformally flat nature of the space–time. As a consequence of the conservation laws a direct relation between the cooling rate per unit volume of matter and the energy density of the neutrino flux as measured by a radially moving observer is obtained. It is also shown that the outward neutrino flux contributes to the contraction of the star. The possibility of occurrence of a 'gravitational bounce' and a consequent oscillatory motion of the fluid particles of the sphere is also pointed out.

1. Introduction

Theoretical studies about the gravitational collapse of massive bodies have attracted considerable attention in recent years. Historically, the problem was first considered by Datt (1938) and by Oppenheimer and Snyder (1939). It was assumed that the imploding object is uniform, spherically symmetric, without rotation and pressure, and initially at rest. The interior geometry of such an object is supposed to be described by the Robertson–Walker cosmological metric. They concluded that when all the thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse freely under the influence of its own gravitational field.

Nearly a quarter of a century later the discovery of star-like objects with intense radio/optical emission which demand energy sources of the order of 10^{60} erg (which is the rest energy of 10^6 suns) led several authors including Bondi (1964), Misner and Sharp (1964), May and White (1966), McVittie (1966), Thompson and Whitrow (1967, 1968) to introducing more physically tenable conditions into this idealized problem.

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In addition to these works recently Vaidya (1968) obtained a class of nonstatic solutions representing fluid spheres and possessing the property that the four dimensional stream-lines are orthogonal to the hypersurfaces ρ = constant. This class includes as particular cases the well-known Schwarzschild solution with constant density and that of Oppenheimer and Snyder (1939).

Since the problem of gravitational collapse is essentially a problem of energy release it is necessary to consider situations where a spherically symmetric star emits during gravitational collapse a non-negligible fraction of its mass as neutrino radiation. Although the dynamics of such a collapse are more complicated, notable contributions in this direction have already been made by Bondi (1964), Misner (1965), Vaidya (1966) and others. In this case the fluid does not obey a simple adiabatic equation of state but each element of the fluid will cool by emission of neutrinos at some rate determined by its temperature and density. The mechanism of neutrino flux is simplified by assuming that all the neutrinos move radially outward when emitted and that they are neither scattered nor absorbed by the surrounding matter.

In the present paper the dynamics of spherical gravitational collapse is very much simplified by introducing the eigenvalue of the conformal Weyl tensor in the 2–2 component of the Einstein tensor. It is known (Krishna Rao 1966) that the Weyl tensor of a general spherically symmetric space–time is of type D in Petrov's classification and therefore, there is only one independent eigenvalue which is real. The 2–2 component of the Einstein tensor for spherically symmetric space–times can be written with the help of this eigenvalue and the geometry takes a very simple form. Also, since the Weyl tensor, having all the symmetry properties of a vacuum Riemann tensor, is to be thought of as representing the *free gravitational field* (Szekeres 1966), its eigenvalue may be interpreted as the energy density of the free gravitational field. It would indeed be seen that this energy density coupled to the material energy density acts in a way similar to the mean density of matter filling the sphere.

2. Description of the gravitational field

To give an analytic description of the gravitational field of a nonrotating star, we must first choose a coordinate system and then give relative to that coordinate system, the metric tensor g_{ab} which determines the geometry of the space–time. We choose the Schwarzschild type of coordinates and the metric immediately takes the form

$$ds^2 = -e^{\lambda}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu}\, dt^2 \tag{1}$$

where λ and ν are functions of r and t only. Using relativistic units ($G = 1, c = 1$) and denoting partial differentiation with respect to r and t by a prime and a dot respectively, the nonvanishing components of the energy–momentum tensor obtained through Einstein's equations are

$$8\pi T_1^1 = r^{-2}(1-e^{-\lambda}) - r^{-1}\, e^{-\lambda}\nu' \tag{2}$$

$$8\pi T_2^2 = 8\pi T_3^3 = 8\pi\epsilon - r^{-2}(1-e^{-\lambda}) + r^{-1}\, e^{-\lambda}(\lambda' - \nu') \tag{3}$$

$$8\pi T_4^4 = r^{-2}(1-e^{-\lambda}) + r^{-1}\, e^{-\lambda}\lambda' \tag{4}$$

$$8\pi T_{14} = r^{-1}\dot{\lambda} \tag{5}$$

where

$$8\pi\epsilon = \tfrac{1}{4}e^{-\nu}\{2\ddot{\lambda}+(\dot{\lambda}-\dot{\nu})\dot{\lambda}\}+(1-e^{-\lambda})r^{-2}-\tfrac{1}{4}e^{-\lambda}\{2\nu''+\nu'(\nu'-\lambda') + 2(\lambda'-\nu')r^{-1}\}$$

is the eigenvalue of the Weyl tensor in Petrov's classification as shown by Krishna Rao (1966). (The factor 8π is introduced for later convenience.) Since ϵ is a scalar, the equations (2)–(5) contain explicitly only first derivatives of λ and ν.

3. The energy–momentum tensor

In order to give a physical significance to the equations (2)–(5), we introduce purely locally minkowskian coordinates (x, y, z, τ) by

$$dx = e^{\lambda/2}\,dr \qquad dy = r\,d\theta \qquad dz = r\sin\theta\,d\phi \qquad d\tau = e^{\nu/2}\,dt$$

λ and ν being treated as constants. Designating the minkowskian components of the energy tensor by a bar, we have

$$\bar{T}^1_1 = T^1_1 \qquad \bar{T}^2_2 = T^2_2 \qquad \bar{T}^3_3 = T^3_3$$
$$\bar{T}^4_4 = T^4_4 \qquad \bar{T}_{14} = T_{14}\exp\{-\tfrac{1}{2}(\lambda+\nu)\}.$$

Next we suppose that when viewed by an observer moving relative to these coordinates with a velocity w in the radial (x) direction, the physical content of the space–time consists of:

(i) an isotropic fluid of density $\hat{\rho}$ and pressure $\hat{p}$;
(ii) isotropic radiation of energy density $3\hat{q}$;
(iii) unpolarized neutrino flux of energy density $\hat{\sigma}$ travelling in the radial direction.

When viewed by this moving observer, the covariant energy–momentum tensor in minkowskian coordinates is thus

$$\begin{bmatrix} \hat{p}+\hat{q}+\hat{\sigma} & 0 & 0 & -\hat{\sigma} \\ 0 & \hat{p}+\hat{q} & 0 & 0 \\ 0 & 0 & \hat{p}+\hat{q} & 0 \\ -\hat{\sigma} & 0 & 0 & \hat{\rho}+3\hat{q}+\hat{\sigma} \end{bmatrix}.$$

A Lorentz transformation readily shows that

$$T^1_1 = \bar{T}^1_1 = -(\rho w^2+p)(1-w^2)^{-1}-\sigma \tag{6}$$

$$T^2_2 = \bar{T}^2_2 = T^3_3 = \bar{T}^3_3 = -p \tag{7}$$

$$T^4_4 = \bar{T}^4_4 = (\rho+pw^2)(1-w^2)^{-1}+\sigma \tag{8}$$

$$T_{14} = \bar{T}_{14}\exp\{\tfrac{1}{2}(\lambda+\nu)\} = -\exp\{\tfrac{1}{2}(\lambda+\nu)\}\{(\rho+p)w(1-w^2)^{-1}+\sigma\} \tag{9}$$

where $\rho = \hat{\rho}+3\hat{q}$, $p = \hat{p}+\hat{q}$, $\sigma = \hat{\sigma}(1+w)/(1-w)$ and $w = \exp\{\tfrac{1}{2}(\lambda-\nu)\}(dr/dt)$. Here σ is the energy density of neutrinos in the rest frame of the fluid and w tells us how the fluid of the sphere moves. Since $\hat{q}$ occurs only in these combinations, we can from now on work with only four quantities ρ, p, σ and w.

Now from (2)–(4) and (6)–(8) we readily obtain

$$e^{-\lambda} = 1-\frac{8\pi}{3}(\rho+\epsilon)r^2. \tag{10}$$

Comparing this expression for $e^{-\lambda}$ with the one obtained by Bondi (1964), we get

$$m(r,t) = \frac{4\pi}{3}(\rho+\epsilon)r^3 \tag{11}$$

where

$$m(r,t) = \int_0^r (4\pi r^2\,dr)T^4_4 \tag{12}$$

is the mass contained within a coordinate radius r of the fluid sphere and the integration being performed for constant t. The expression for $m(r,t)$ given by (11) shows that $(\rho+\epsilon)$ plays the part of the mean density of matter within a coordinate radius r of the sphere. From (10) it can be seen that a singularity at the origin $r = 0$ may be avoided only when $(\rho+\epsilon)_{r=0}$ goes as a power of r higher than inverse square. The coupling of ϵ with the material energy density ρ suggests that the former may be interpreted as the energy density of the free gravitational field. In the case when the paths of the fluid particles are geodesics, the force distribution of this free gravitational field has the effect of distorting a sphere into an ellipsoid which has the r direction as the principal axis and is degenerate in the θ, ϕ plane (Szekeres 1965). When $\epsilon = 0$, the Weyl tensor vanishes identically and the space–time is conformally flat. The space–time describing the nonstatic generalizations of the Schwarzschild interior solution obtained by Vaidya (1968) is conformally flat. We shall discuss this case in detail in the later sections. Now the equation (2) may be used to determine ν on each t = constant hypersurface.

By a slight rearrangement of (9) and substituting from (4), (5) and (10), the energy equation takes the form

$$\frac{D}{Dt}\left(\frac{4\pi}{3}(\rho+\epsilon)r^3\right) = -4\pi r^2(p-\sigma)\frac{dr}{dt}-4\pi r^2\exp\{\tfrac{1}{2}(\nu-\lambda)\}\sigma \tag{13}$$

where

$$\frac{D}{Dt} \equiv \frac{dr}{dt}\frac{\partial}{\partial r}+\frac{\partial}{\partial t}$$

is the differentiation operator following the fluid. This shows how the total pressure (fluid and radiation) does work on a material sphere across its moving boundary and as Bondi pointed out if one remembers that $\exp\{\frac{1}{2}(\nu-\lambda)\}$ is the velocity of light in our coordinate system, the last term gives the neutrino flux over a two sphere of coordinate radius r. The advantage of (13) over similar expressions given by Bondi (1964) and by Misner (1965) is that the part played by the energy density of the free gravitational field during collapse has been brought to notice.

Note that we have a full set of four equations for the determination of the four physical variables ρ, p, σ and w. For any given pair of values of λ and ν we can write the physical variables as

$$8\pi\rho = -8\pi\epsilon+3r^{-2}(1-e^{-\lambda}) \tag{14}$$

$$8\pi p = -8\pi\epsilon+r^{-2}(1-e^{-\lambda})-r^{-1}e^{-\lambda}(\lambda'-\nu') \tag{15}$$

$$8\pi\sigma = \frac{e^{-\lambda}(e^{-\lambda}\lambda'\nu' - e^{-\nu}\dot{\lambda}^2) - 4r^{-2}(1 - e^{-\lambda} - 4\pi\epsilon r^2)\{1 - e^{-\lambda} - 4\pi\epsilon r^2 + \frac{1}{2}r\, e^{-\lambda}(\nu' - \lambda')\}}{r\, e^{-\lambda/2}\{(e^{-\lambda/2}\lambda' + e^{-\nu/2}\dot{\lambda}) + (e^{-\lambda/2}\nu' + e^{-\nu/2}\dot{\lambda})\}} \tag{16}$$

$$w = \frac{2(1 - e^{-\lambda} - 4\pi\epsilon r^2) - e^{\lambda/2} r(e^{-\lambda/2}\lambda' + e^{-\nu/2}\dot{\lambda})}{2(1 - e^{-\lambda} - 4\pi\epsilon r^2) + e^{-\lambda/2} r(e^{-\lambda/2}\nu' + e^{-\nu/2}\dot{\lambda})}. \tag{17}$$

The totality of phenomena that can be described by (14) to (17) is very large. However, if we wish to find λ and ν explicitly we must assume two relations between pressure, density and the cooling rate per unit amount of matter. Any explicit interior solution thus obtained can be matched with Vaidya's (1951, 1953) exterior radiating Schwarzschild solution over a moving boundary say, $r = a(u)$, u being the retarded time.

4. The metric form for the case $\epsilon = 0$

For the discussion of spherically symmetric material distributions in the absence of the free gravitational field ($\epsilon = 0$), it is convenient to choose the metric in the conformally flat form. The advantages of such a choice are (i) the light geometry inside a star is that of the flat Minkowski continuum, (ii) relativistic stellar structure at least in its kinematical aspect forms a link between special relativity and gravitation, and (iii) the number of unknown functions in the metric tensor reduces to one and hence it is easy to handle. Further, imposing the condition of spatial isotropy the four dimensional riemannian manifold describing the interior geometry of a star is given by the metric

$$\begin{aligned} ds^2 &= \alpha^{-2}(R, T)\eta_{ab}\, dx^a\, dx^b \\ \eta_{ab}\, dx^a\, dx^b &\equiv dT^2 - dR^2 - R^2(d\theta^2 + \sin^2\theta\, d\phi^2) \end{aligned} \tag{18}$$

where R, θ, ϕ are the spherical polar coordinates of euclidean space.

It should be noted here that it is quite unnecessary to suppose that the star is spatially homogeneous in the sense that matter at every depth has the same thermodynamic functions. In fact the diffusion of neutrinos from the centre of the star towards the outer layers makes any assumption of homogeneity inappropriate.

The nonvanishing components of the energy–momentum tensor T_a^b for the metric (18) obtained through Einstein's field equations are

$$\begin{aligned} 8\pi T_1^1 &= 4\alpha\frac{\alpha_1}{R} - 3\alpha_1^2 - 2\alpha\alpha_{44} + 3\alpha_4^2 \\ 8\pi T_2^2 &= 8\pi T_3^3 = 2\alpha(\alpha_{11} - \alpha_{44}) - 3(\alpha_1^2 - \alpha_4^2) + 2\alpha\frac{\alpha_1}{R} \\ 8\pi T_4^4 &= 2\alpha\alpha_{11} + 4\alpha\frac{\alpha_1}{R} - 3(\alpha_1^2 - \alpha_4^2) \\ 8\pi T_4^1 &= -8\pi T_1^4 = -2\alpha\alpha_{14} \end{aligned} \tag{19}$$

where the subscripts 1 and 4 after α denote differentiation with respect to R and T respectively.

By an appropriate modification of the scheme given in § 3 and using (19) we immediately obtain

$$8\pi\rho = 3\left(2\alpha\frac{\alpha_1}{R}-\alpha_1^2+\alpha_4^2\right) \tag{20}$$

$$8\pi p = 2\alpha(\alpha_{44}-\alpha_{11})-3(\alpha_4^2-\alpha_1^2)-2\alpha\frac{\alpha_1}{R} \tag{21}$$

$$4\pi\sigma = \frac{\alpha[\{\alpha_{11}-(\alpha_1/R)\}\{\alpha_{44}+(\alpha_1/R)\}-\alpha_{14}^2]}{\alpha_{11}+2\alpha_{14}+\alpha_{44}} \tag{22}$$

$$w = -\frac{\alpha_{11}-(\alpha_1/R)+\alpha_{14}}{\alpha_{44}+(\alpha_1/R)+\alpha_{14}}. \tag{23}$$

5. Conservation laws

It is convenient, for the discussion of conservation laws, to write the energy–momentum tensor given in § 3 as

$$T^{ab} = M^{ab}+N^{ab} \tag{24}$$

$$M^{ab} = (\rho+p)u^a u^b - pg^{ab}$$
$$u_a u^a = 1 \qquad \frac{u^1}{u^4} = w \qquad u^2 = 0 \qquad u^3 = 0 \tag{25}$$

$$N^{ab} = \sigma k^a k^b \qquad k_a k^a = 0. \tag{26}$$

Here M^{ab} and N^{ab} are respectively the matter (perfect fluid) and neutrino energy–momentum tensors. The matter conservation law is expressed in the form of the equation of continuity

$$(nu^a)_{;a} = 0 \tag{27}$$

where n is the baryon number density. Since the neutrino emission decreases the internal energy of matter, following Misner (1965), we introduce the cooling rate per unit amount of matter $C(\mathscr{T}, n)$, $\mathscr{T}$ being the temperature. Then nC is the cooling rate per unit volume in the rest frame of the fluid and we can write the equation governing the neutrino flux as

$$-u_a M^{ab}{}_{;b} = nC = u_a N^{ab}{}_{;b} \tag{28}$$

since the total stress–energy tensor T^{ab} satisfies the local energy and momentum conservation laws

$$T^{ab}{}_{;b} = 0. \tag{29}$$

As a consequence of (29), through (28), we get

$$(\rho+p)u^a{}_{;b}u^b = (g^{ab}-u^a u^b)p_{,b}-N^{ab}{}_{;b}+nCu^a. \tag{30}$$

In view of

$$k_a N^{ab} = 0 \qquad k_{a;b}N^{ab} = 0$$

we get the identity

$$k_a N^{ab}{}_{;b} = 0. \tag{31}$$

In the present case $-k_1 = k_4, k_2 = k_3 = 0$, and it follows from (31) that

$$N^{1b}{}_{;b} = N^{4b}{}_{;b}. \tag{32}$$

Therefore, we get from (28) by making use of (32)

$$N^{1b}{}_{;b} = \alpha n C\left(\frac{1+w}{1-w}\right)^{1/2}. \tag{33}$$

Of the four equations in (30) two (for $a = 2, 3$) are satisfied identically and the remaining two after making use of (32) and (33) combine into a single equation which reads as

$$\begin{aligned}(\rho+p)\{(ww_{,1}+w_{,4})(1-w^2)^{-1}-\alpha^{-1}(\alpha_1+w\alpha_4)\}\\ = -p_{,1}-wp_{,4}-\alpha^{-1}nC(1-w^2)^{1/2}.\end{aligned} \tag{34}$$

In a comoving coordinate system ($w = 0$) this equation of hydrodynamics looks like a simple hydrostatic balance of forces.

6. Slow collapse

For slow gravitational collapse the neutrino production vanishes and we write β in place of α and an overhead suffix 'o' for ρ, p, w in the foregoing analysis. The differential equation for β is readily obtained from $\sigma = 0$

$$\left(\beta_{11}-\frac{\beta_1}{R}\right)\left(\beta_{44}+\frac{\beta_1}{R}\right)-\beta_{14}^2 = 0. \tag{35}$$

A first integral of this equation is

$$\frac{\beta_1}{R}T+\beta_4 = f\left(\frac{\beta_1}{R}\right) \tag{36}$$

f being arbitrary. It may be pointed out here that the cosmological solutions of Infeld and Schild (1945) as well as Schwarzschild's interior solution are particular solutions of (36).

In view of (35) the expression for $\mathring{w}$ written from (23) takes a very simple form

$$\mathring{w} = -\frac{\beta_{11}-(\beta_1/R)+\beta_{14}}{\beta_{44}+(\beta_1/R)+\beta_{14}} = -\frac{\beta_{14}}{\beta_{44}+(\beta_1/R)} = -\frac{(\beta_1/R)_{,1}}{(\beta_1/R)_{,4}}. \tag{37}$$

Now it can be verified that:

(i) the stream lines of the fluid are orthogonal to the hypersurfaces $\mathring{\rho}(R, T) =$ constant, that is

$$\mathring{\rho}_{,1}+\mathring{w}\mathring{\rho}_{,4} = 0. \tag{38}$$

(ii) In view of (37) and (38) the conservation equation for matter written from (34) as

$$(\mathring{\rho}+\mathring{p})\beta^{-1}(\mathring{w}\mathring{w}_{,1}+\mathring{w}_{,4})(1-\mathring{w}^2)^{-1}+\{(\mathring{\rho}+\mathring{p})\beta^{-1}\}_{,1}+\mathring{w}\{(\mathring{\rho}+\mathring{p})\beta^{-1}\}_{,4} = 0 \tag{39}$$

turns out to be an identity. (This can also be seen from the fact that we have four independent components of the energy–momentum tensor for the determination of the four unknown quantities $\mathring{\rho}$, $\mathring{p}$, $\mathring{w}$ and β.)

Finally, the rate of contraction of the fluid sphere is given by

$$\mathring{U} = \mathring{u}^a \frac{\partial(R/\beta)}{\partial x^a} = \frac{R}{\beta}\left\{\left(\frac{\beta}{R} - \beta_1\right)\mathring{w} - \beta_4\right\}(1-\mathring{w}^2)^{-1/2}.$$

From the expression for $\mathring{\rho}$ it may be readily verified that

$$\frac{\mathring{\rho}_{,1}}{\beta_{14}} = \frac{\mathring{\rho}_{,4}}{\beta_{44}+(\beta_1/R)} = -3\frac{\beta(1-w^2)^{1/2}}{4\pi R}\mathring{U}$$

and hence

$$\mathring{U} = 0 \Leftrightarrow \mathring{\rho} = \text{constant}.$$

7. Collapse with neutrino emission

It is easy to see from (6)–(9) after making the necessary modifications that σ does not enter the expression for $w(=(T^1_1 - T^4_1 - T^2_2)/(T^4_4 + T^4_1 - T^2_2))$ and hence without loss of generality we assume that $w = \mathring{w}$. Therefore

$$\frac{\alpha_{11}-(\alpha_1/R)+\alpha_{14}}{\alpha_{44}+(\alpha_1/R)+\alpha_{14}} = \frac{\beta_{11}-(\beta_1/R)+\beta_{14}}{\beta_{44}+(\beta_1/R)+\beta_{14}} \tag{40}$$

where β satisfies (36). From (40) the relation between α and β is given by

$$\alpha = \beta + F + RF' \tag{41}$$

F being an arbitrary function of the retarded time $u \equiv T - R$ and a prime on F denotes differentiation with respect to u. Hence we have the following result.

Given any perfect fluid solution of Einstein's field equations with a conformally flat metric of the form

$$ds^2 = \beta^{-2}\eta_{ab}\, dx^a\, dx^b \tag{42}$$

one can always include the presence of a neutrino energy–momentum tensor, $\sigma k_a k_b$, $k_a k^a = 0$, without disturbing the conformally flat nature of the space–time. That is, the new metric is given by

$$ds^2 = (\beta + F + RF')^{-2}\eta_{ab}\, dx^a\, dx^b. \tag{43}$$

Thus, $F(u)$ being arbitrary, for any given β one obtains a family of solutions. It may be mentioned here that the analytic solutions for gravitational collapse with radiation obtained by Vaidya (1966) are particular cases of the above general result (Patel 1969).

When expressions (20) and (21) are written in terms of F and β, by making use of (41), one notices that

$$\frac{\alpha}{\rho + p} = \frac{\beta}{\mathring{\rho} + \mathring{p}} \tag{44}$$

and σ takes a very simple form

$$4\pi\sigma = R(\beta + F + RF')F'''. \tag{45}$$

In view of (37)–(39), (44), (45) the conservation equation (34) turns out to be a direct relation between the cooling rate per unit volume of matter nC and the energy density of neutrino flux $\hat{\sigma}$ as measured by a radially moving observer

$$nC = 3\left(\frac{F}{R}+\frac{\beta}{R}-\beta_1-\beta_4\right)\hat{\sigma}\left(\frac{1+w}{1-w}\right)^{1/2}. \tag{46}$$

Also the rate of contraction of the fluid sphere is

$$U = u^a\frac{\partial(R/\alpha)}{\partial x^a} = \frac{\beta}{\alpha}\mathring{U}-\left(\frac{R}{\alpha}\right)^2 u^a\frac{\partial\{(F/R)+F'\}}{\partial x^a}. \tag{47}$$

To understand the nature of $u^a\partial\{(F/R)+F'\}/\partial x^a$ we consider a star which is initially static, for example the Schwarzschild interior solution, for which $\mathring{U} = 0$. Then

$$U = -\left(\frac{R}{\alpha}\right)^2 u^a\frac{\partial\{(F/R)+F'\}}{\partial x^a}$$

and remembering that $4\pi(R/\alpha)^2$ is the surface area of the 2-space $R = \text{constant}$, $T = \text{constant}$ (which is indistinguishable from an ordinary sphere of radius R/α), the presence of the factor $(R/\alpha)^2$ on the right hand side of the above expression is significant. Now for contraction, $U < 0$ (for expansion $U > 0$), $u^a\partial\{(F/R)+F'\}/\partial x^a > 0$ (<0). It may be mentioned here that if $\{(F/R)+F'\}$ is a periodic function 'gravitational bounce would occur'.

Now in the general case for contraction to occur, $U < 0$

$$\mathring{U}-\frac{R^2}{\alpha\beta}u^a\frac{\partial\{(F/R)+F'\}}{\partial x^a} < 0 \tag{48}$$

and three cases arise:

(i) $\qquad \mathring{U} < 0 \qquad u^a\dfrac{\partial\{(F/R)+F'\}}{\partial x^a} > 0.$

Here both the material and neutrino forces contribute for contraction and the inequality (48) always holds good. Thus once the contraction sets in it continues till the star degenerates or shrinks into a point singularity.

(ii) $\qquad \mathring{U} > 0 \qquad u^a\dfrac{\partial\{(F/R)+F'\}}{\partial x^a} > 0.$

In this case the neutrino forces which favour contraction dominate over the material expansive forces.

(iii) $\qquad \mathring{U} < 0 \qquad u^a\dfrac{\partial\{(F/R)+F'\}}{\partial x^a} < 0.$

In this case the inequality (48) shows that if a star is contracting initially, it will continue to do so in spite of the expansive forces created by neutrino emission.

8. Conclusion

The introduction of the eigenvalue of the Weyl tensor into the scheme of spherical symmetry leads to a simpler geometrical outlook of the whole problem. The absence of second derivatives of the metric potentials should prove helpful in numerical computations. Also by avoiding evaluation of integrals for the mass $m(r, t)$ given by (12) some possible inconsistency that may arise in the initial value problem is eliminated. It will be shown elsewhere that the procedure adopted here is useful for the discussion of contraction of charged fluid spheres.

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